Lecture 3: ODEs + Differential/Vector Calculus

ODEs

- First, we will use some basic ODE solution methods.

ex.) $y' = xy$ $\to$ $\frac{y'}{y} = x$ $\xrightarrow{\text{integrate}}$ $\ln|y| = \frac{1}{2}x^2 + C$

$\to$ $y = Ce^{x^2/2}$

- this is Separation of Variables. In general,

$\frac{dy}{dt} = g(y)h(t)$ and we solve $\int \frac{dy}{g(y)} = \int h(t)dt$

- Higher-order ODEs are usually reduced to a system of first-order ODEs.

e.g.) $y'' = -k^2 y$ has solution $y(t) = c_1 e^{ikt} + c_2 e^{-ikt}$
(by the auxilliary eqn.)

or, introduce $w = (y, y')$ and $w' = \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix} w$

we could then diagonalize and solve to get the same answer.

- Another method to solve first-order ODE systems is Picard iteration.

Assume we have the problem

$$\frac{dw}{dt} = F(t, w) \quad \& \quad w(t_0) = w_0 \qquad (B)$$

we form a sequence

& $u_0(t) = w_0$

$u_{m+1}(t) = w_0 + \int_{t_0}^{t} F(s, u_m(s))ds$

$\leadsto$ under certain assumptions, $u_{n+1}(t)$ approach a solution:

$\boxed{\text{Th}^{\underline{m}}}$ Suppose $F \in C^0(I \times U)$ where I is an open interval containing t_0 and U is a domain in $\mathbb{R}^n$ containing w_0.
Assume $F(t_0, x) \in C^1(U)$ for any fixed $t_0 \in I$. Then, (B) admits a unique solution on $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$.

Picard-Lindelöf

Applying this to the example, $u_0 = w_0 = (a, b)$,

$u_1(t) = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} b(t-t_0) \\ -k^2(t-t_0) \end{pmatrix}$ and for $t_0 = 0$.

$\vdots$ $\quad$ > Show this as an exercise!

$u_k(t) = \sum_{j=0}^{k} \frac{t^j}{j!} \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix}^j \begin{pmatrix} a \\ b \end{pmatrix}$

$u(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix}^j \begin{pmatrix} a \\ b \end{pmatrix}$ Solves the problem!

Vector Calculus

- Since we're in vector space, we must frame our calculus this way (Math 53/54)
  - Recall for $f \in C^1(\mathcal{U})$, $\nabla f = (\partial_{x_1} f, \partial_{x_2} f, \dots \partial_{x_n} f)$ (gradient)
    and for $v \in C^1(\mathcal{U}, \mathbb{R}^n)$
    $\nabla \cdot v = \sum_{i=1}^n \partial_{x_i} v_i$ (divergence)
  - The Laplacian: $\Delta u = \cancel{\nabla(\nabla(u))}\ \nabla \cdot (\nabla u) = \sum_{i=1}^n \partial_{x_i}^2 u$

- Leibniz Integral Rule:
  If $\mathcal{U} \subset \mathbb{R}^n$ is a bounded domain, u & $\partial u / \partial t \in C^0((a,b) \times \mathcal{U})$
  then $\frac{d}{dt} \int_{\mathcal{U}} u(t,x)\, dx = \int_{\mathcal{U}} \frac{\partial u}{\partial t}(t,x)\, dx$

- We will use surface integrals over our domain. It is not often we will directly compute them, but you should know what the integral means.
  For example, let ~~$\mathcal{U} = S^{n-1}$, the unit~~ $\mathcal{U} = B(0,1)$ so $\partial \mathcal{U} = S^{n-1}$, the unit sphere.

$$\int_{\partial \mathcal{U}} f \underbrace{dS}_{\text{"surface integral"}} = \int_V f(\sigma(w)) \left| \det \left[\frac{\partial \sigma}{\partial w_1}, \dots, \frac{\partial \sigma}{\partial w_{n-1}}, \vec{\eta} \right] \right| dw$$

where $\sigma : V \to \partial \mathcal{U}$ is a parameterization of $\partial \mathcal{U}$ & $\vec{\eta}$ the unit normal to $\partial \mathcal{U}$.

- For the unit sphere, $\vec{\eta}(x) = x$ as pictured: 
  Then, if σ is a parameterization of S^{n-1}, we may use the change-of-coordinates $x = r \cdot \sigma(y)$ to integrate spherically

$$dx = \left| \det \left[\frac{\partial x}{\partial y_1} \cdots \frac{\partial x}{\partial y_{n-1}}, \frac{\partial x}{\partial r} \right] \right| dr\, dy_1 \dots dy_{n-1}$$
$$= r^{n-1} dr\, dS(y)$$

> good exercise in definition

So $\int_{B(0;R)} f(x)\, dx = \int_{S^{n-1}} \int_0^R f(r\sigma(y))\, r^{n-1} dr\, dS(y)$

$\boxed{Th^{\underline{m}}}$ The Divergence theorem: Suppose $U \subset \mathbb{R}^n$ is a bounded domain with piecewise-C^1 boundary. For a vector field $F \in C^1(\bar{U}; \mathbb{R}^n)$

$$\int_U \nabla \cdot F \, dx = \int_{\partial U} F \cdot \vec{\eta} \, dS$$

for outward unit normal $\vec{\eta}$ to ∂U.

- Since the Laplacian $\Delta u = \nabla \cdot (\nabla u)$, we may set $F = \nabla u$ and

$$\int_U \Delta u \, dx = \int_{\partial U} (\nabla u) \cdot \vec{\eta} \, dS = \int_{\partial U} \frac{\partial u}{\partial \eta} \, dS \qquad (C)$$

ex.) $U = B(0,a)$. Assume we wish to integrate a radial function $g(r)$ for $r = |x|$. As above,

$$\int_{B(0,a)} \Delta g \, dx = \int_{S^{n-1}} \int_0^a (\Delta g(r)) \, r^{n-1} \, dr \, dS \quad \text{> Fubini \& since } g \text{ is radial.}$$

$$= \int_0^a (\Delta g(r)) \, r^{n-1} \, dr \; \mathrm{Vol}(S^{n-1})$$

By formula (C),

$$(D) \qquad \int_0^a \Delta g(r) \, r^{n-1} \, dr = a^{n-1} \frac{\partial g}{\partial r}(a) \qquad \text{as follows.}$$

First, g is radial and $\mathrm{Vol}(\partial B(0,a)) = a^{n-1} \mathrm{Vol}(S^{n-1})$,

so $$\left[\frac{\partial g}{\partial r}(a)\right] a^{n-1} = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\partial B(0,a)} \frac{\partial g}{\partial r}(a) \, dS = \frac{1}{\mathrm{Vol}(S^{n-1})} \int \frac{\partial g}{\partial \eta} \, dS$$

because $\eta = \frac{x}{a}$ implies $\frac{\partial g}{\partial \eta} = \frac{\partial g}{\partial r}$.

Second,

$$\frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\partial B(0,a)} \frac{\partial g}{\partial \eta} \, dS = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{B(0,a)} \Delta g \, dx = \int_0^a \Delta g(r) \, r^{n-1} \, dr.$$

If we differentiate (D) with respect to a, we compute the Laplacian radially

$$a^{n-1} \Delta g(a) = \frac{\partial}{\partial a}\left[a^{n-1} \frac{\partial g}{\partial r}(a)\right]$$

or

$$\Delta g(a) = a^{1-n} \frac{\partial}{\partial a}\left[a^{n-1} \frac{\partial g}{\partial r}(a)\right]$$

• Lastly, we have Green's Identities.

If $U \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary, then for $u \in C^2(\bar{U})$ and $v \in C^1(\bar{U})$,

$$\int_U \nabla v \cdot \nabla u + v \Delta u \, dx = \int_{\partial U} v \frac{\partial u}{\partial \eta} \, dS$$

if $v \in C^2(\bar{U})$,

$$\int_U v \Delta u - u \Delta v \, dx = \int_{\partial U} v \frac{\partial u}{\partial \eta} - u \frac{\partial v}{\partial \eta} \, dS$$

Pf: Set $F = v \nabla u$ and $\nabla \cdot F = \nabla v \cdot \nabla u + v \Delta u$. Applying the divergence theorem gives the first identity.
Swapping the roles of u and v, and subtracting, gives the second.